Assembly Strategies for Remanufacturing Systems with Variable Quality Return

Xiaoning Jin, S. Jack Hu, Jun Ni, and Guoxian Xiao

In this supplemental material, proofs of sufficient conditions for optimality, proofs of optimal policy structure and proofs of monotonicity of the properties in the paper are presented.

Proof Lemma 1

Condition (A1) implies that if it is optimal to satisfy a demand using substitutable inventory $x_{12}$ in state $V(0, x_{21}, x_{12}, x_{22})$, i.e.,

$$V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) + c_i + c_s < V(x_{11}, x_{21}, x_{12}, x_{22}) + s_i,$$

it is also optimal to satisfy a demand by assembly with substitution in state $V(0, x_{21}, x_{12}, x_{22} + l)$, i.e.,

$$V(x_{11}, x_{21} - l, x_{12} - l, x_{22} + l) + c_i + c_s < V(x_{11}, x_{21}, x_{12}, x_{22} + l) + s_i.$$

Thus reassembly with substitution decision policy follows a threshold policy where the threshold for switching decision is determined as $\gamma_{22}(x) \cap \min\{x_{22} \mid \Delta_{low}(x) \leq 0\}$.

Condition (A2) indicates that if $V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) + c_i + c_s > V(x_{11}, x_{21}, x_{12}, x_{22}) + s_i$ (optimal not to fulfill demand with substitutable module assembly in state $V(0, x_{21}, x_{12}, x_{22})$), then it is also optimal not to fulfill demand with substitutable module assembly in state

$$V(0, x_{21} + l, x_{12} + l, x_{22})$$

i.e., $V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s > V(x_{11}, x_{21} + l, x_{12} + l, x_{22}) + s_i$, thus implying a threshold for the assembly with substitution control.

Similarly, Condition A3 implies the assembly with substitution optimal decision is also constrained by a combinational effect of $x_{21}, x_{12}$ and $x_{22}$ such that if is optimal to assembly with
substitution in state \( V(0, x_{21}, x_{12}, x_{22}) \), then it is also optimal to assemble with substitution in state \( V(0, x_{21} + 1, x_{12} + 1, x_{22} + 1) \).

**Proof of Lemma 2**

The proof is similar and hence omitted.

**Proof of Lemma 3**

In our model, we formulate the controlled queueing systems using dynamic programming framework which encounters a set of properties and operations. Recent work by Benjaafar et al. (2008) establishes preservation results for an assembly system with multiple stages. Our model requires similar sub/super-modularity conditions, but our analysis extends to a multi-dimensional case due to the arbitrary number of quality levels and types of modules which prevent us from using their results directly.

Given Lemma 1 and Lemma 2 and Definition 1 and Definition 2, we know that the thresholds conditions B1-B4 are implied by conditions A1-A4, respectively. It only remains to show that Conditions A1 through A4 are preserved under the operators \( T^{(1)} \) and \( T^{(2)} \).

**Condition A1**

\((B1) \ \Delta_{low} V(x) \geq \Delta_{low} V(x + e_{22})\)

The difference operator remains the same: \( \Delta_{low} V(x) = V(x - e_{12} - e_{21}) - V(x) \)

For the ease of exposition, we use the up/down arrows \( \uparrow \) and \( \downarrow \) to represent how \( \Delta_{low} V \) changes with system state. For example, \( \Delta_{low} V(x) \downarrow x_{22} \) represents that the marginal cost to assemble with substitution in state \( x \), \( \Delta_{low} V \), decreases when there is an additional inventory \( x_{22} \).
Now from condition A1, we have $\Delta_{low} V(x) \downarrow x_{22}$ and we need to show that $\Delta_{low} T^{(1)} V(x) \downarrow x_{22}$ to prove that property A1 preserves under operator $T^{(1)}$

That is equivalent to show:

$$\Delta_{low} T^{(1)} V(x + e_{22}) - \Delta_{low} T^{(1)} V(x) \leq 0$$

$$\Delta_{low} T^{(1)} V(x + e_{22}) = \min \{ V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} + 1) + c_i + c_s, V(x_{11}, x_{21}, x_{12}, x_{22} + 1) + s_i \}$$

$$- \min \{ V(x_{11}, x_{21}, x_{12}, x_{22} + 1) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) + s_i \}$$

$$\Delta_{low} T^{(1)} V(x) = \min \{ V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) + c_i + c_s, V(x_{11}, x_{21}, x_{12}, x_{22}) + s_i \}$$

$$- \min \{ V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) + s_i \}$$

Now we need to show (1)-(2) $\leq 0$, there are four outcomes of (1):

Case1: $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} + 1)$

Case2: $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) + c_i + c_s - s_i$

Case3: $V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - (c_i + c_s - s_i)$

Case4: $V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1)$

Let $\tilde{c} = c_i + c_s - s_i$

h) Suppose case1 is the outcome of (1), and there are 3 possible outcomes of (2):

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22})$ is the outcome of (1), then (1)-(2) becomes

  $$\Delta_{low} V(x + e_{22}) - \Delta_{low} V(x) \leq 0$$

  by condition A1.

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + \bar{c}$ is the outcome of (2), then (1)-(2)

  $$\leq \Delta_{low} V(x + e_{22}) - \Delta_{low} V(x) \leq 0$$

  by condition A1.

- If $V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + \bar{c}$ is the outcome of (2), then (1)-(2) =

  $$\Delta_{low} V(x + e_{22}) - \Delta_{low} V(x + e_{21} + e_{12}) \leq 0$$

  by condition A2.
i) Suppose case 2 is the outcome of (1), and there are 3 possible outcomes of (2):

- If  
  $$V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) 
  $$\leq \Delta_{low} V(x + e_{22}) - \Delta_{low} V(x) \leq 0$$  
  because of property A1.

- If  
  $$V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) - V(x_{11}, x_{21} + l, x_{12} + l, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) 
  $$= \Delta_{low} V(x + e_{22}) - \Delta_{low} V(x) + \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x + e_{21} + e_{12}) \leq 0$$

- If  
  $$V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + l, x_{12} + l, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) because of  
  $$\leq \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x + e_{21} + e_{12}) \leq 0$$  
  condition A1.

j) Suppose case 4 is the outcome of (1), and there are three possible outcomes of (2):

- If  
  $$V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) is 
  $$\leq \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$$  
  because of condition A3.

- If  
  $$V(x_{11}, x_{21} - l, x_{12} - l, x_{22}) - V(x_{11}, x_{21} + l, x_{12} + l, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) is 
  $$= \Delta_{low} (x + e_{21} + e_{12} + e_{22}) - \Delta_{low} (x) \leq 0$$  
  because of condition A3.

- If  
  $$V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + l, x_{12} + l, x_{22}) + \tilde{c}$$  
  is the outcome of (2), then (1)-(2) is 
  $$= \Delta_{low} (x + e_{21} + e_{12} + e_{22}) - \Delta_{low} (x + e_{21} + e_{12}) \leq 0$$  
  because of condition A1.

Now we have shown that condition A1 is preserved under operator \( T^{(1)} \).

**Condition A2**

(B2)  
$$\Delta_{low} V(x) \leq \Delta_{low} V(x + e_{21} + e_{12})$$

The difference operator remains the same:  
$$\Delta_{low} V(x) = V(x - e_{12} - e_{21}) - V(x)$$

We have A2  
$$\Delta_{low} V(x) \uparrow (x_{21}, x_{12})$$  
and we need to show that \( \Delta_{low} T^{(1)} V(x) \uparrow (x_{21}, x_{12}) \) to prove that property A2 preserves under operator \( T^{(1)} \).
That is equivalent to show: \( \Delta_{low} T^{(i)} V(x + e_{21} + e_{12}) - \Delta_{low} T^{(i)} V(x) \leq 0 \)

\[
\Delta_{low} T^{(i)} V(x + e_{21} + e_{12}) = \min \{ V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) + s_i \} \\
- \min \{ V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + c_i + c_s, V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22}) + s_i \}
\]

\[
\Delta_{low} T^{(i)} V(x) = \min \{ V(x_{11}, x_{11} - 1, x_{12} - 1, x_{12}) + c_i + c_s, V(x_{11}, x_{11}, x_{12}, x_{22}) + s_i \} \\
- \min \{ V(x_{11}, x_{11}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) + s_i \} \quad (3)
\]

\[ \Delta_{low} T^{(i)} V(x) = \min \{ V(x_{11}, x_{11}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) + s_i \} \quad \ldots \ldots \quad (4) \]

Now we need to show (3)-(4) \( \geq 0 \), there are four outcomes of (3):

Case1: \( V(x_{11}, x_{11}, x_{12}, x_{22}) - V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) + c_i + c_s - s_i \)

Case2: \( V(x_{11}, x_{11}, x_{12}, x_{22}) - V(x_{11}, x_{11} + 2, x_{12} + 2, x_{22}) + c_i + c_s - s_i \)

Case3: \( V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) - V(x_{11}, x_{11} + 2, x_{12} + 2, x_{22}) + c_i + c_s - s_i \)

Let \( \tilde{c} = c_i + c_s - s_i \)

a) Suppose case1 is the outcome of (3), and there are 3 possible outcomes of (4):

- If \( V(x_{11}, x_{11} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{11}, x_{12}, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4) = \( \Delta_{low} V(x + e_{21} + e_{12}) - \Delta_{low} V(x) \geq 0 \) because of condition A2.

- If \( V(x_{11}, x_{11} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4) = \( V(x_{11}, x_{11} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{11} + 1, x_{12} + 1, x_{22}) \)

b) Suppose case2 is the outcome of (3), and there are 3 possible outcomes of (4):
If \( V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4)

\[
\geq \Delta_{low} V(x + e_{21} + e_{12}) - \Delta_{low} V(x) \geq 0 \text{ because of condition A2.}
\]

If \( V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21}, +1, x_{12} + 1, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4)

\[
= V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22})
\]
\[
-V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) + V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22})
\]
\[
\geq V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22})
\]
\[
= 0
\]

If \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4)

\[
= V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22})
\]
\[
-V(x_{11}, x_{21}, x_{12}, x_{22}) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22})
\]
\[
\geq V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22})
\]
\[
= 0
\]

c) Suppose Case 3 is the outcome of (3), and there are 3 possible outcomes of (4):

If \( v(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - v(x_{11}, x_{21}, x_{12}, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4)

\[
= V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22})
\]
\[
-V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) + V(x_{11}, x_{21}, x_{12}, x_{22})
\]
\[
\geq \Delta_{low} V(x + 2e_{21} + 2e_{12}) - \Delta_{low} V(x)
\]
\[
= \Delta_{low} V(x + 2e_{21} + 2e_{12}) - \Delta_{low} V(x + e_{21} + e_{12})
\]
\[
+ \Delta_{low} V(x + e_{21} + e_{12}) - \Delta_{low} V(x) \geq 0
\]

because of condition A2.

If \( V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21}, +1, x_{12} + 1, x_{22}) + \tilde{c} \) is the outcome of (4), then (3)-(4)

\[
= V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22}) - V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22})
\]
\[
+V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22})
\]
\[
\geq \Delta_{low} V(x + 2e_{21} + 2e_{12}) - \Delta_{low} V(x + e_{21} + e_{12}) \geq 0
\]

because of condition A2.
• If \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + \bar{c} \) is the outcome of (4), then (3)-(4)
\[
\begin{align*}
&= V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22}) \\
&- V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) \\
&= \Delta_{low} V(x + 2e_{21} + 2e_{12}) - \Delta_{low} V(x + e_{21} + e_{12}) \geq 0
\end{align*}
\]
because of condition A2.

**Condition A3**

(A3) \( \Delta_{low} V(x) \geq \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) \)

The difference operator remains the same: \( \Delta_{low} V(x) = [V(x - e_{12} - e_{21}) + c_r] - [V(x) + s_1] \)

We have Condition A3 \( \Delta_{low} V(x) \downarrow (x_{21}, x_{12}, x_{22}) \) and we need to show that
\( \Delta_{low} T^{(1)} V(x) \downarrow (x_{21}, x_{12}, x_{22}) \) to prove that property A3 preserves under operator \( T^{(1)} \)

That is equivalent to show:
\[ \Delta_{low} T^{(1)} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} T^{(1)} V(x) \leq 0 \]

\[
\begin{align*}
\Delta_{low} T^{(1)} V(x + e_{21} + e_{12} + e_{22}) \\
= \min \{ V(x_{11}, x_{21}, x_{12}, x_{22} + 1) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) + s_1 \} \\
- \min \{ V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) + c_i + c_s, V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} + 1) + s_1 \}
\end{align*}
\]
\[
\begin{align*}
\Delta_{low} T^{(1)} V(x) = \min \{ V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) + c_i + c_s, V(x_{11}, x_{21}, x_{12}, x_{22}) + s_1 \} \\
- \min \{ V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + s_1 \}
\end{align*}
\]
\[
\begin{align*}
\Delta_{low} T^{(1)} V(x) = \min \{ V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + s_1 \} \\
- \min \{ V(x_{11}, x_{21}, x_{12}, x_{22}) + c_i + c_s, V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + s_1 \}
\end{align*}
\]

Now we need to show (5)-(6) \( \geq 0 \), there are four outcomes of (5):

Case1: \( V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) \)

Case2: \( V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} + 1) + c_i + c_s - s_1 \)

Case3: \( V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} + 1) \)

Let \( \bar{c} = c_i + c_s - s_1 \)
a) Suppose case 1 is the outcome of (5), and there are three possible outcomes of (6):

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} ) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $= \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$ because of condition A3.

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} ) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $= V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} ) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) \leq V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) = \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x + e_{21} + e_{12}) \leq 0$ because of condition A1.

- If $V(x_{11}, x_{21}, x_{12}, x_{22} ) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $= V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) \leq V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) = \Delta_{low} V(x + e_{21} + e_{12} + e_{22}) - \Delta_{low} V(x + e_{21} + e_{12}) \leq 0$ because of condition A1.

b) Suppose case 2 is the outcome of (5), and there are 3 possible outcomes of (6):

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} ) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $\leq \Delta_{low} V(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$ because of condition A3.

- If $V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22} ) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $\leq V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} ) \leq V(x_{11}, x_{21}, x_{12}, x_{22} + 1) - V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} + 1) - V(x_{11}, x_{21}, x_{12}, x_{22} ) + V(x_{11}, x_{21} + 2, x_{12} + 2, x_{22} ) = \Delta_{low} V(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$.

- If $V(x_{11}, x_{21}, x_{12}, x_{22} ) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $\leq \Delta_{low} V(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$ because of condition A3.

- If $V(x_{11}, x_{21}, x_{12}, x_{22} ) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} ) + \tilde{c}$ is the outcome of (6), then (5)-(6) $\leq \Delta_{low} V(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low} V(x) \leq 0$ because of condition A1 and A3.

c) Suppose case 3 is the outcome of (5), and there are 3 possible outcomes of (6):
If \( V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22}) + \tilde{c} \) is the outcome of (6), then (5)-(6)
\[
\Delta_{low, V}(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low, V}(x) \leq 0
\]

If \( V(x_{11}, x_{21} - 1, x_{12} - 1, x_{22}) - V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22}) + \tilde{c} \) is the outcome of (6), then (5)-(6)
\[
\Delta_{low, V}(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low, V}(x) \leq 0
\]

If \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + \tilde{c} \) is the outcome of (6), then (5)-(6)
\[
= \Delta_{low, V}(x + 2e_{21} + 2e_{12} + e_{22}) - \Delta_{low, V}(x + e_{21} + e_{12}) \leq 0
\]
\[= 0 \text{ because of condition A1 and A3.}
\]

Now we have shown that condition A1, A2 and A3 are preserved under operator \( T^{(1)} \).

**Condition A4**

\[(A4) \ \Delta_{high, V}(x) \leq \Delta_{high, V}(x + e_{12} + e_{22}) \iff \Delta_{high, V}(x) \uparrow (x_{12}, x_{22})\]

For the higher demand class, the difference operator is
\[
\Delta_{high, V}(x) \uparrow [V(x - e_{12} - e_{22}) + 2c_2] - [V(x) + s_2],
\]
and we need to show that conditions A4 are preserved under operator \( T^{(2)} \).

We have \( \Delta_{high, V}(x) \uparrow (x_{12}, x_{22}) \) and we need to show that \( \Delta_{high, T^{(2)}, V}(x) \uparrow (x_{12}, x_{22}) \) to prove that property A4 preserves under operator \( T^{(2)} \).

That is equivalent to show: \( \Delta_{high, T^{(2)}, V}(x + e_{12} + e_{22}) - \Delta_{high, T^{(2)}, V}(x) \geq 0 \)
\[
\Delta_{high, T^{(2)}, V}(x + e_{12} + e_{22}) = \min \{V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + s_2 \}
\]
\[
- \min \{V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + 2c_2, V(x_{11}, x_{21}, x_{12} + 2, x_{22} + 2) + s_2 \}
\]
\[
= \min \{V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) + 2c_2, V(x_{11}, x_{21}, x_{12} + x_{22}) + s_2 \}
\]
\[- \min \{V(x_{11}, x_{21}, x_{12}, x_{22}) + 2c_2, V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + s_2 \}\]
\]
\[
\Delta_{high, T^{(2)}, V}(x) = \min \{V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) + 2c_2, V(x_{11}, x_{21}, x_{12} + x_{22}) + s_2 \}
\]
\[
- \min \{V(x_{11}, x_{21}, x_{12}, x_{22}) + 2c_2, V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + s_2 \}\]
\]
\[
\Delta_{high, T^{(2)}, V}(x) \geq 0 \text{ for } (7)-(8) \geq 0,
\]
\[
\text{there are two feasible outcomes of (7):}
\]
Case 1: \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22} + 1) \)

Case 2: \( V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) - V(x_{11}, x_{21}, x_{12} + 2, x_{22} + 2) \)

Case 3: \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12} + 2, x_{22} + 2) + (2c_2 - s_2) \)

Let \( \hat{c} = 2c_2 - s_2 \)

a) Suppose case 1 is the outcome of (7), and there are 2 possible outcomes of (8):

- If \( V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) - V(x_{11}, x_{21}, x_{12}, x_{22}) \) is the outcome of (8), then (7)-(8)

\[ \geq \Delta_{\text{high}} V(x + e_{12} + e_{22}) - \Delta_{\text{high}} V(x) \geq 0 \] because of condition A4.

- If \( V(x_{11}, x_{21}, x_{12}, x_{22}) - V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) - \hat{c} \) is the outcome of (8), then (7)-(8)

\[ \geq \Delta_{\text{high}} V(x + e_{12} + e_{22}) - V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) - 2c_2 \\
+ V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + 2c_2 \\
= \Delta_{\text{high}} V(x + e_{12} + e_{22}) - \Delta_{\text{high}} V(x) = 0 \text{ by condition A4.} \]

b) Suppose Case 2 is the outcome of (7), and there are two possible outcomes of (8):

- If \( V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) + 2c_2 - V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) - s_2 \) is the outcome of (8), then (7)-(8)

\[ \geq \Delta_{\text{high}} V(x + 2e_{12} + 2e_{22}) - V(x_{11}, x_{21}, x_{12}, x_{22}) - s_2 + V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) + s_2 \\
= \Delta_{\text{high}} V(x + 2e_{12} + 2e_{22}) - \Delta_{\text{high}} V(x + e_{12} + e_{22}) \geq 0 \text{ by condition A4.} \]

- If \( V(x_{11}, x_{21}, x_{12}, x_{22}) + s_2 - V(x_{11}, x_{21}, x_{12} + 1, x_{22} + 1) - s_2 \) is the outcome of (8), then (7)-(8)

\[ = \Delta_{\text{high}} V(x + 2e_{12} + 2e_{22}) - \Delta_{\text{high}} V(x + e_{12} + e_{22}) \geq 0 \text{ by condition A4.} \]

c) Suppose Case 3 is the outcome of (7), and there are two possible outcomes of (8):

- If \( V(x_{11}, x_{21}, x_{12} - 1, x_{22} - 1) + 2c_2 - V(x_{11}, x_{21}, x_{12}, x_{22}) - 2c_2 \) is the outcome of (8), then (7)-(8)
\[ \Delta_{\text{high}} V(x + e_{12} + e_{22}) - \Delta_{\text{high}} V(x) \geq 0 \text{ by condition A4.} \]

- If \( V(x_{11}, x_{21}, x_{12}, x_{22}) + s_2 - V(x_{11}, x_{21}, x_{12}, x_{22}) - 2c_2 \) is the outcome of (8), then (7)-(8)

\[
\begin{align*}
\Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}) &\leq \Delta_{\text{low}} V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) \\
\Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}) &\leq \Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}) + 2c_2 \\
\Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}) &\leq \Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}) + 2c_2 \\
\Delta_{\text{low}} V(x + e_{12} + e_{22}) &+ \Delta_{\text{low}} V(x + 2e_{12} + 2e_{22}) \geq 0 \\
\end{align*}
\]

Now we have shown that Condition A4 is preserved under operator \( T^{(2)} \).

**Proof of Lemma 4**

**L4.1:** From condition A2 and the definition of \( \gamma_{22}(x) \), we have

\[
\Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}, \gamma_{22}(x + e_{21} + e_{12})) \leq \Delta_{\text{low}} V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1) \leq 0. 
\]

Hence, we have \( \Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22}, \gamma_{22}(x + e_{21} + e_{12})) \leq 0 \). Using the definition of \( \gamma_{22}(x) \), we can obtain

\[
\gamma_{22}(x) \leq \gamma_{22}(x + e_{21} + e_{12}).
\]

**L4.2:** We know that \( \Delta_{\text{low}} V(x_{11}, x_{21}, x_{12}, x_{22} + 1 \mid x_{21} + x_{12} = \kappa(x + e_{22})) \geq 0 \) because of (8) in Definition 3.

From condition A1, we have that \( \Delta_{\text{low}} V(x_{11}, x_{21} + 1, x_{12} + 1, x_{22} + 1 \mid x_{21} + x_{12} = \kappa(x + e_{22})) \geq 0 \). Using the definition of \( \kappa(x) \), we obtain that \( \kappa(x) \leq \kappa(x + e_{22}) \).

The proof of Lemma 5 is very similar, and so it is omitted. □
Proof of Theorem 1

Consider a value iteration algorithm to solve the dynamic programming for the optimal control problem given in (2.5) where initial values $V_0(x) = 0$ are used for each state $x$. Conditions B1-B4 are trivially satisfied by $V_0(x)$, hence $V_0(x) = 0$. We apply $V_{t+1}(x) = TV_t(x)$ for $t = 0,1,2,...$ to determine the relative value functions for sequential iterations. Suppose $V_t(x) \in \Omega$, then, Lemma 1 and Lemma 2 show that $V_{t+1}(x)$ also satisfy condition B1-B4, i.e., $V_{t+1}(x) \in \Omega$. Furthermore, the problem given in (2.5) and assumes that the total rates of receiving returned modules are less than the total demand rates for modules, the inventories of various module types and quality grades are finite and bounded. The action set of the problem consists of finite number of actions, i.e., for lower class of demand: (a) reassemble with substitution, and (b) do not reassemble but use new ones; for higher class of demand: (c) reassemble and (d) do not reassemble but use new ones. Thus, the problem is a finite state, finite action set problem. In addition, the underlying Markov chain is also unichain. Therefore, the existence of a long-run average cost and the validity of the value iteration algorithm are ensured by Theorem 8.4.5 of Puterman (1994).